Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M **16/01**

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2016

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EIGENVALUE PLACEMENT FOR REGULAR MATRIX PENCILS WITH RANK ONE PERTURBATIONS

HANNES GERNANDT* AND CARSTEN TRUNK†

Abstract. A regular matrix pencil $sE - A$ and its rank one perturbations are considered. We determine the sets in $\mathbb{C} \cup \{\infty\}$ which are the eigenvalues of the perturbed pencil. We show that the largest Jordan chains at each eigenvalue of $sE - A$ may disappear and the sum of the length of all destroyed Jordan chains is the number of eigenvalues (counted with multiplicities) which can be placed arbitrarily in $\mathbb{C} \cup \{\infty\}$. We prove sharp upper and lower bounds of the change of the algebraic and geometric multiplicity of an eigenvalue under rank one perturbations. Finally we apply our results to a pole placement problem for a single-input differential algebraic equation with feedback.

Key words. regular matrix pencils, rank one perturbations, matrix spectral perturbation theory

AMS subject classifications. 15A22, 15A18, 47A55, 15A29

1. Introduction. Let E, A be square matrices of size $n \times n$ with complex entries. We consider rank one perturbations of matrix pencils of the form

$$\mathcal{A}(s) := sE - A, \quad s \in \mathbb{C}. \tag{1.1}$$

The pencil $\mathcal{A}$ is assumed to be *regular*, meaning that $\det(sE - A) \not\equiv 0$ holds. The corresponding spectral theory is a generalization of the eigenvalue problem for matrices [11, 22] which has a lot of applications. In this note we investigate how the spectrum of a regular matrix pencil can be changed under perturbations of rank one. For this we investigate the maximal change of the Jordan chains under rank one perturbations. This was already studied in [2, 9, 21]. In [9] it was shown that for regular matrix pencils under low-rank perturbations generically only the largest Jordan chain at each eigenvalue is destroyed, see also [1, 18]. From the perturbation bounds for the Jordan chains at each eigenvalue we derive sharp upper and lower bounds on the algebraic and geometric multiplicities at each eigenvalue under rank one perturbations. These bounds only depend on the unperturbed pencil. Now in contrast to the results obtained in [1, 9, 18], our investigation focuses also on the placeability of the spectrum of regular matrix pencils under rank one perturbations. The main result states that a certain number of eigenvalues (counted with multiplicities), satisfying the obtained bounds on the multiplicities, can be placed arbitrarily in $\overline{\mathbb{C}}$ under a perturbation of rank one. In particular we also obtain an analogue result for real-valued matrices E, A under real valued rank one perturbations.
In special cases this placement is considered in the literature. For E positive definite and A symmetric the placement problem was studied in [10]. In the matrix case, i.e. $E = I_n$, a special case of the eigenvalue placement problem was studied in [16] and for symmetric A in [13]. Our main method to solve the placement problem for regular matrix pencils is an investigation of the perturbation determinant, which was also considered in [14, 19, 20, 21]. Our results have various applications: Our research is motivated by an eigenvalue placement problem arising in the design of electrical circuits (cf. [3, 14]). Another application discussed, is given by the pole assignment problem for single input differential-algebraic equations. This problem is well studied

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†Various Affiliations, supported by various foundation grants.

even for singular matrix pencils and multi input systems, see [6] and the references therein. However, for single input systems, we obtain simple conditions on the number of poles that can be assigned arbitrarily.

2. Eigenvalues and Jordan Chains of Matrix Pencils. In this section the notion of eigenvalues and Jordan chains for matrix pencils $\mathcal{A}(s) = sE - A$ with $E, A \in \mathbb{C}^{n\times n}$ is recalled. Furthermore we summarize some basic spectral properties which are implied by the well known Weierstraß canonical form (cf. [11]).
For fixed $\lambda \in \mathbb{C}$ observe that $\mathcal{A}(\lambda)$ is a matrix over $\mathbb{C}$. Hence the *spectrum* of $\mathcal{A}$ is defined as

$$\sigma(\mathcal{A}) := \{\lambda \in \mathbb{C} \mid 0 \text{ is an eigenvalue of } \mathcal{A}(\lambda)\}, \quad \text{if } E \text{ is invertible,}$$

and

$$\sigma(\mathcal{A}) := \{\lambda \in \mathbb{C} \mid 0 \text{ is an eigenvalue of } \mathcal{A}(\lambda)\} \cup \{\infty\}, \quad \text{if } E \text{ is singular.}$$

Obviously the spectrum of $\mathcal{A}$ is a subset of the extended complex plane $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and the roots of the *characteristic polynomial* $\det(sE - A)$ are exactly the elements of $\sigma(\mathcal{A}) \setminus \{\infty\}$. Hence the spectrum consists of finitely many points iff $\mathcal{A}$ is regular. For $\mathcal{A}$ singular one always has $\sigma(\mathcal{A}) = \overline{\mathbb{C}}$.
We recall the notion for Jordan chains and root subspaces (cf. [12, Section 1.4], [17, §11.2]). The set $\{g_0, \dots, g_{m-1}\} \subset \mathbb{C}^n$ is a *Jordan chain* of *length* m at $\lambda \in \mathbb{C}$ if $g_0 \neq 0$ and

$$(A - \lambda E)g_0 = 0, \quad (A - \lambda E)g_1 = Eg_0, \quad \dots, \quad (A - \lambda E)g_{m-1} = Eg_{m-2} \tag{2.1}$$

and we call $\{g_0, \dots, g_{m-1}\} \subset \mathbb{C}^n$ a Jordan chain of length m at ∞ if

$$g_0 \neq 0, \quad Eg_0 = 0, \quad Eg_1 = Ag_0, \quad \dots, \quad Eg_{m-1} = Ag_{m-2}. \tag{2.2}$$

Two Jordan chains $\{g_0, \dots, g_k\}$ and $\{h_0, \dots, h_l\}$ at $\lambda \in \overline{\mathbb{C}}$ are called *linearly independent*, if the vectors $g_0, \dots g_k, h_0, \dots, h_l$ are linearly independent. Furthermore, we say that $\mathcal{A}$ has k Jordan chains of length m if there exist k linearly independent Jordan chains of length m at $\lambda \in \overline{\mathbb{C}}$. We denote for $\lambda \in \overline{\mathbb{C}}$ and $l \in \mathbb{N} \setminus \{0\}$ the subspace of all elements of all Jordan chains up to the length l at λ by

$$\mathcal{L}^l_\lambda(\mathcal{A}) := \Big\{ g_j \in \mathbb{C}^n \mid 0 \leq j \leq l-1, \ \{g_0, \dots, g_j\} \text{ is a Jordan chain at } \lambda \Big\}$$

and the root subspace which consists of all elements of all Jordan chains at λ,

$$\mathcal{L}_\lambda(\mathcal{A}) := \bigcup_{l=1}^{\infty} \mathcal{L}^l_\lambda(\mathcal{A}).$$

It is well known that regular pencils $sE - A$ can be transformed into the Weierstraß canonical form (cf. [11, Chapter XII, §2]), i.e. there exist invertible matrices $S, T \in \mathbb{C}^{n\times n}$ and $r \in \{0, 1, \dots, n\}$ such that

$$S(sE - A)T = s\begin{pmatrix} I_r & 0 \\ 0 & N \end{pmatrix} - \begin{pmatrix} J & 0 \\ 0 & I_{n-r} \end{pmatrix}, \tag{2.3}$$

with $J \in \mathbb{C}^{r\times r}$ and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ in Jordan canonical form and N nilpotent. From the Weierstraß canonical form, we can deduce some well known properties (cf. [4, 11]).

LEMMA 2.1. *Let $\mathcal{A}$ be regular with Weierstraß canonical form* (2.3), *then the following holds.*

(a) *A Jordan chain* $\{g_0, \ldots, g_{m-1}\}$ *of* $\mathcal{A}$ *at* $\lambda \in \mathbb{C}$ *of length* m *corresponds to a Jordan chain* $\{\pi_r T^{-1} g_0, \ldots, \pi_r T^{-1} g_{m-1}\} \subset \mathbb{C}^r$ *of* J *at* λ *of length* m. *Here* π_r *denotes the projection of* $x \in \mathbb{C}^n$ *onto the first* r *entries. Vice versa a Jordan chain* $\{h_0, \ldots, h_{m-1}\}$ *of* J *at* λ *corresponds to a Jordan chain* $\left\{T\begin{pmatrix} h_0 \\ 0 \end{pmatrix}, \ldots, T\begin{pmatrix} h_{m-1} \\ 0 \end{pmatrix}\right\}$ *of* $\mathcal{A}$ *at* λ.

(b) *A Jordan chain* $\{g_0, \ldots, g_{m-1}\}$ *of* $\mathcal{A}$ *at* ∞ *of length* m *corresponds to a Jordan chain* $\{\pi_{n-r} T^{-1} g_0, \ldots, \pi_{n-r} T^{-1} g_{m-1}\} \subset \mathbb{C}^{n-r}$ *of* N *at* 0 *of length* m. *Here* π_{n-r} *denotes the projection of* $x \in \mathbb{C}^n$ *onto the last* $n-r$ *entries. Vice versa a Jordan chain* $\{h_0, \ldots, h_{m-1}\}$ *of* N *at* 0 *corresponds to a Jordan chain* $\left\{T\begin{pmatrix} 0 \\ h_0 \end{pmatrix}, \ldots, T\begin{pmatrix} 0 \\ h_{m-1} \end{pmatrix}\right\}$ *of* $\mathcal{A}$ *at* ∞.

(c) *We have* $\sigma(\mathcal{A}) \backslash \{\infty\} = \sigma(J)$ *and the characteristic polynomial of* $\mathcal{A}$ *is divisible by the minimal polynomial* m_J *of* J *with*

$$\det(sE - A) = (-1)^{n-r} \det(ST)^{-1} m_J(s) q(s),$$

where q *is a monic polynomial of degree* $r - \deg m_J$. *The value* $\lambda \in \sigma(\mathcal{A}) \backslash \{\infty\}$ *is a root of* q *if and only if* $\dim \ker \mathcal{A}(\lambda) \geq 2$. *Moreover the multiplicity of a root* λ *of* $\det(sE - A)$ *is equal to* $\dim \mathcal{L}_\lambda(\mathcal{A})$ *and we have*

$$\sum_{\lambda \in \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A}) = n. \tag{2.4}$$

For each eigenvalue $\lambda \in \sigma(\mathcal{A})$ we set

$$k(\lambda) := \dim \ker \mathcal{A}(\lambda)$$

and, by Proposition 1, this corresponds to the number of linearly independent Jordan chains of J or N at λ. Each of these $k(\lambda)$ different Jordan chains has a length which we denote by $m_j(\lambda)$, $1 \leq j \leq k(\lambda)$. These numbers $m_j(\lambda)$ are not uniquely determined, more precisely, they depend on the chosen Weierstraß canonical form (2.3) and they are unique up to permutations. In the following, we will choose those numbers in a specific way and we fix this in the following assumption.

ASSUMPTION 2.2. *Given a regular pencil* $sE - A$ *which has Weierstraß canonical form* (2.3) *with* $r \in \{0, 1, \ldots, n\}$ *and matrices* $J \in \mathbb{C}^{r \times r}$ *and* $N \in \mathbb{C}^{(n-r) \times (n-r)}$. *Then we assume that for* $\lambda \in \sigma(\mathcal{A})$ *the numbers* $m_j(\lambda)$, $1 \leq j \leq k(\lambda)$, *are sorted in a non-decreasing order*

$$m_1(\lambda) \geq \ldots \geq m_{k(\lambda)}(\lambda). \tag{2.5}$$

Observe that Assumption 2.2 is no restriction for regular pencils. This means that for every regular pencil the matrices S and T in (2.3) can be chosen in such a way that the Jordan blocks of J satisfy the condition (2.5) (cf. [11]). With Assumption 2.2 the minimal polynomial m_J of J can be written as

$$m_J(s) = \prod_{\lambda \in \sigma(J)} (s - \lambda)^{m_1(\lambda)}. \tag{2.6}$$

3. The structure of rank one pencils. In this section we study pencils of rank one. Recall that the rank of a pencil $\mathcal{A}$ is the largest $r \in \mathbb{N}$, such that $\mathcal{A}(s)$, viewed as a matrix with polynomial entries, has minors of size r, that are not the zero polynomial (cf. [9, 11]). This implies that $\mathcal{A}$ has rank equal to n if and only if it regular. Hence, pencils of rank one are not regular for $n \geq 2$, meaning that they cannot be transformed to Weierstraß canonical form. Nevertheless, there is a simple representation given in the Proposition below.

PROPOSITION 3.1. *The pencil $\mathcal{P}(s) = sF - G$ with $F, G \in \mathbb{C}^{n \times n}$ has rank one if and only if there exists $u, v, w \in \mathbb{C}^n$ with $w \neq 0$ and $u \neq 0$ or $v \neq 0$ such that*

$$\mathcal{P}(s) = (su + v)w^* \quad or \quad \mathcal{P}(s) = w(su^* + v^*). \tag{3.1}$$

For $F, G \in \mathbb{R}^{n \times n}$ there exist $u, v, w \in \mathbb{R}^n$ such that (3.1) *holds.*

Proof. Given that $\mathcal{P}$ has rank one, then by definition all minors of $\mathcal{P}(s)$ of size strictly larger than one vanish for all $s \in \mathbb{C}$. This implies

$$\operatorname{rk} \mathcal{P}(s) = \operatorname{rk}(sE - A) \leq 1, \quad \text{for all } s \in \mathbb{C}. \tag{3.2}$$

From (3.2) for $s = 0$, we see $\operatorname{rk} A \leq 1$, so there exist $u, v \in \mathbb{C}^n$ with $A = uv^*$. For $s = 1$ in equation (3.2), we obtain $\operatorname{rk}(E - A) \leq 1$ so there exists $w, z \in \mathbb{C}^n$ with $E - A = wz^*$. Using the representations above we see

$$2E - A = 2(E - A) + A = 2wz^* + uv^*.$$

From (3.2) for $s = 2$ we conclude again that $\operatorname{rk}(2E - A) \leq 1$. Let us first consider the case that u and w are linearly independent. From the rank condition we conclude $z = \alpha v$ or $v = \alpha z$ for some $\alpha \in \mathbb{C}$. Let $z = \alpha v$ (the case $v = \alpha z$ can be proven similarly), then

$$sE - A = s(uv^* + wz^*) - uv^* = (s(u + \alpha w) - u)v^*,$$

therefore $\mathcal{P}$ admits a representation as in (3.1). Assume u and w are linearly dependent. Let $u = \beta w$ for some $\beta \in \mathbb{C}$ (the case $w = \beta u$ can be proven similarly), then

$$sE - A = s(uv^* + wz^*) - uv^* = w(s(\beta v^* + z^*) - \beta v^*)$$

holds, hence (3.1) is proven. The converse statement is obvious. For $E, A \in \mathbb{R}^{n \times n}$ the arguments above remain valid after replacing $\mathbb{C}$ by $\mathbb{R}$. ◻

The following example illustrates that both representations in (3.1) are necessary.

EXAMPLE 3.2. *A short computation shows that the matrix pencils*

$$\begin{aligned} \mathcal{P}_1(s) &:= \begin{pmatrix} s+1 & s+1 \\ 1 & 1 \end{pmatrix} = \left(s \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) (1, 1), \\ \mathcal{P}_2(s) &:= \begin{pmatrix} s+1 & 1 \\ s+1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (s(1, 0) + (1, 1)) \end{aligned}$$

admit only one of the representations given in Proposition 3.1. In the case where in (3.1) the elements $u, v \in \mathbb{C}^n$ are linearly dependent, there exist non-zero $(\alpha, \beta) \in \mathbb{C}^2$ such that

$$\mathcal{P}(s) = (\alpha s - \beta)uw^* \quad \text{or} \quad \mathcal{P}(s) = (\alpha s - \beta)vw^*. \tag{3.3}$$

this means that both representations in (3.1) coincide. The next lemma provides a simple criterion for $\mathcal{A} + \mathcal{P}$ to be regular when $\mathcal{P}$ is of the form (3.3).

LEMMA 3.3. *Let $\mathcal{A}(s) = sE - A$ be regular and choose $(\alpha, \beta) \in \mathbb{C}^2$ non-zero and let $\mathcal{P}$ be given by* (3.3). *Then the following holds.*

(a) *Assume* $\alpha \neq 0$. *If* $\beta/\alpha \in \sigma(\mathcal{A})$ *then* $\beta/\alpha \in \sigma(\mathcal{A}+\mathcal{P})$ *for all* $u, v \in \mathbb{C}^n$. *If* $\beta/\alpha \notin \sigma(\mathcal{A})$ *then* $\mathcal{A}+\mathcal{P}$ *is regular for all* $u, v \in \mathbb{C}^n$.
(b) *Assume* $\alpha = 0$. *If* $\infty \in \sigma(\mathcal{A})$ *then* $\infty \in \sigma(\mathcal{A}+\mathcal{P})$ *for all* $u, v \in \mathbb{C}^n$. *If* $\infty \notin \sigma(\mathcal{A})$ *then* $\mathcal{A}+\mathcal{P}$ *is regular for all* $u, v \in \mathbb{C}^n$.

Proof. For $\alpha \neq 0$ we look at the equation

$$\det(\mathcal{A}+\mathcal{P})(\beta/\alpha) = \det\left(\frac{\beta}{\alpha}E - A + (\alpha\frac{\beta}{\alpha} - \beta)uv^*\right) = \det\left(\frac{\beta}{\alpha}E - A\right) = \det\mathcal{A}(\beta/\alpha).$$

From this we see that the statements in (a) hold true. For $\alpha = 0$ we use that $\infty \in \sigma(\mathcal{A})$ if and only if the leading coefficient E has full rank equal to n. Since we assume $\alpha = 0$, the leading coefficient of $\mathcal{A}+\mathcal{P}$ is E for all $u, v \in \mathbb{C}^n$, hence (b) is proven. □

4. Change of the root subspaces under rank one perturbations. In this section, we obtain bounds on the number of eigenvalues which can be changed by a rank one perturbation. In the following we will combine [9, Lemma 2.1] with Proposition 2.1.

LEMMA 4.1. *Let* $\mathcal{A}$ *satisfy Assumption 2.2 and let* $\mathcal{P}$ *be of rank one. Assume that* $\mathcal{A}+\mathcal{P}$ *is regular and let* λ *be an eigenvalue of* $\mathcal{A}$. *Then* $\mathcal{A}+\mathcal{P}$ *has at least* $k(\lambda)-1$ *linearly independent Jordan chains at* λ *of length* $\tilde{m}_i(\lambda)$ *such that*

$$\tilde{m}_2(\lambda) \geq \ldots \geq \tilde{m}_{k(\lambda)}(\lambda) \quad \textit{and} \quad \tilde{m}_i(\lambda) \geq m_i(\lambda), \quad 2 \leq i \leq k(\lambda).$$

The following result describes the maximal change of the root subspace dimension under rank one perturbations. For matrices, that is, if $E = I_n$, this result was obtained in [21], see also [2].

PROPOSITION 4.2. *Let* $\mathcal{A}$ *satisfy Assumption 2.2, then for any rank one pencil* $\mathcal{P}$ *such that* $\mathcal{A}+\mathcal{P}$ *is regular we have for all* $\lambda \in \overline{\mathbb{C}}$ *and* $k \in \mathbb{N}\setminus\{0\}$

$$\left|\dim\frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{k}(\mathcal{A}+\mathcal{P})} - \dim\frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A})}{\mathcal{L}_\lambda^{k}(\mathcal{A})}\right| \leq 1, \tag{4.1}$$

$$|\dim\mathcal{L}_\lambda^{k}(\mathcal{A}+\mathcal{P}) - \dim\mathcal{L}_\lambda^{k}(\mathcal{A})| \leq k. \tag{4.2}$$

Proof. We prove the inequality (4.1). Assume $\lambda \neq \infty$ and that for $k, l \in \mathbb{N}\setminus\{0\}$ we have

$$\dim\frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A})}{\mathcal{L}_\lambda^{k}(\mathcal{A})} = l. \tag{4.3}$$

This is equivalent to the fact $\mathcal{A}$ has l linearly independent Jordan chains at λ with length at least $k+1$ which means

$$m_1(\lambda) \geq m_2(\lambda) \ldots \geq m_l(\lambda) \geq k+1.$$

It follows from Lemma 4.1 that $\mathcal{A}+\mathcal{P}$ has at least $l-1$ linearly independent Jordan chains with lengths

$$\tilde{m}_2(\lambda) \geq \ldots \geq \tilde{m}_l(\lambda) \geq m_l(\lambda) \geq k+1$$

which leads to

$$\dim\frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{k}(\mathcal{A}+\mathcal{P})} \geq l-1. \tag{4.4}$$

It remains to show that the expression in (4.4) is less or equal then $l+1$. Indeed, assume

$$\dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{k}(\mathcal{A}+\mathcal{P})} \geq l+2,$$

and replace in the above arguments $\mathcal{A}$ by $\mathcal{A}+\mathcal{P}$ and $\mathcal{A}+\mathcal{P}$ by $\mathcal{A}$ then we obtain

$$\dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A})}{\mathcal{L}_\lambda^{k}(\mathcal{A})} \geq l+1,$$

a contradiction and (4.1) is proved. Now we show (4.2). For $k=1$ the definition of $\mathcal{L}_\lambda^i(\mathcal{A})$ implies $\mathcal{L}_\lambda^1(\mathcal{A}) = \ker \mathcal{A}(\lambda)$. Since $\mathcal{A}(\lambda)$ and $(\mathcal{A}+\mathcal{P})(\lambda)$ are matrices and $\mathcal{P}(\lambda)$ is a matrix of rank at most one (see Lemma 3.1), the dimension formula leads to

$$\begin{aligned} &|\dim \ker \mathcal{A}(\lambda) - \dim \ker (\mathcal{A}+\mathcal{P})(\lambda)| \\ &= \big|n - \dim \operatorname{ran} \mathcal{A}(\lambda) - \big(n - \dim \operatorname{ran} (\mathcal{A}+\mathcal{P})(\lambda)\big)\big| \leq 1. \end{aligned} \tag{4.5}$$

Therefore (4.2) holds for $k=1$. For $k \geq 2$ we have the identity

$$\dim \mathcal{L}_\lambda^k(\mathcal{A}) = \dim \ker \mathcal{A}(\lambda) + \sum_{m=1}^{k-1} \dim \frac{\mathcal{L}_\lambda^{m+1}(\mathcal{A})}{\mathcal{L}_\lambda^{m}(\mathcal{A})}$$

using (4.5) and (4.1) leads to

$$\begin{aligned} &|\dim \mathcal{L}_\lambda^k(\mathcal{A}) - \dim \mathcal{L}_\lambda^k(\mathcal{A}+\mathcal{P})| \\ &\leq |\dim \ker \mathcal{A}(\lambda) - \dim \ker (\mathcal{A}+\mathcal{P})(\lambda)| + \sum_{m=1}^{k-1} \left| \dim \frac{\mathcal{L}_\lambda^{m+1}(\mathcal{A})}{\mathcal{L}_\lambda^{m}(\mathcal{A})} - \dim \frac{\mathcal{L}_\lambda^{m+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{m}(\mathcal{A}+\mathcal{P})} \right| \leq k \end{aligned}$$

□

For $k=1$ the inequality (4.2) leads to the following statement.

COROLLARY 4.3. *Let $\mathcal{A}$ be regular, then for arbitrary $\mathcal{P}$ of rank one we have*

$$\left\{ \lambda \in \sigma(\mathcal{A}) \;\middle|\; \dim \ker \mathcal{A}(\lambda) \geq 2 \right\} \subseteq \sigma(\mathcal{A}+\mathcal{P})$$

and for every $\mu \in \sigma(\mathcal{A}+\mathcal{P}) \setminus \sigma(\mathcal{A})$

$$\dim \ker (\mathcal{A}+\mathcal{P})(\mu) = 1, \tag{4.6}$$

i.e., in this case, there is only one Jordan chain of length $\dim \mathcal{L}_\mu(\mathcal{A}+\mathcal{P})$.

Proposition 4.2 states, roughly speaking, that the largest possible change in the dimensions of $\mathcal{L}_\lambda(\mathcal{A}+\mathcal{P})$ compared with $\mathcal{L}_\lambda(\mathcal{A})$ is bounded by the length of the largest Jordan chain of $\mathcal{A}$ and $\mathcal{A}+\mathcal{P}$. However, it is the aim of Theorem 4.4 below to give bounds for the change of dimension of $\mathcal{L}_\lambda(\mathcal{A}+\mathcal{P})$ only in terms of the unperturbed pencil $\mathcal{A}$. For this, we use the number $m_1(\lambda)$ which is length of the largest Jordan chain of $\mathcal{A}$ at λ, cf. Assumption 2.2, and the number

$$M(\mathcal{A}) := \sum_{\mu \in \sigma(\mathcal{A})} m_1(\mu). \tag{4.7}$$

Theorem 4.4. *Let $\mathcal{A}$ satisfy Assumption 2.2. Then for any rank one pencil $\mathcal{P}$ such that $\mathcal{A}+\mathcal{P}$ is regular we have for $\lambda \in \sigma(\mathcal{A})$*

$$\dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda) \leq \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) \leq \dim \mathcal{L}_\lambda(\mathcal{A}) + M(\mathcal{A}) - m_1(\lambda), \tag{4.8}$$

whereas the change in the dimension for $\lambda \in \overline{\mathbb{C}} \setminus \sigma(\mathcal{A})$ is bounded by

$$0 \leq \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) \leq M(\mathcal{A}). \tag{4.9}$$

Summing up, we obtain the following bounds

$$\begin{aligned} \sum_{\lambda\in\sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) &\geq n - M(\mathcal{A}), \\ \sum_{\lambda\in\sigma(\mathcal{A}+\mathcal{P})\setminus\sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) &\leq M(\mathcal{A}). \end{aligned} \tag{4.10}$$

Proof. By Assumption 2.2, we have $\mathcal{L}_\lambda(\mathcal{A}) = \mathcal{L}_\lambda^{m_1(\lambda)}(\mathcal{A})$. Then (4.2) implies for $\lambda \in \sigma(\mathcal{A})$

$$\begin{aligned} \dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda) &= \dim \mathcal{L}_\lambda^{m_1(\lambda)}(\mathcal{A}) - m_1(\lambda) \\ &\leq \dim \mathcal{L}_\lambda^{m_1(\lambda)}(\mathcal{A}+\mathcal{P}) \leq \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}). \end{aligned} \tag{4.11}$$

This is the lower bound in (4.8). Since $\mathcal{A}+\mathcal{P}$ is regular we can apply (2.4) and (4.11), so the upper bound for $\lambda \in \overline{\mathbb{C}}$ follows from

$$\begin{aligned} \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) &= n - \sum_{\mu\in\sigma(\mathcal{A}+\mathcal{P})\setminus\{\lambda\}} \dim \mathcal{L}_\mu(\mathcal{A}+\mathcal{P}) \\ &\leq \sum_{\mu\in\sigma(\mathcal{A})} \dim \mathcal{L}_\mu(\mathcal{A}) - \sum_{\mu\in\sigma(\mathcal{A})\setminus\{\lambda\}} \dim \mathcal{L}_\mu(\mathcal{A}+\mathcal{P}) \\ &\leq \dim \mathcal{L}_\lambda(\mathcal{A}) + \sum_{\mu\in\sigma(\mathcal{A})\setminus\{\lambda\}} \dim \mathcal{L}_\mu(\mathcal{A}) - \sum_{\mu\in\sigma(\mathcal{A})\setminus\{\lambda\}} \dim \mathcal{L}_\mu(\mathcal{A}+\mathcal{P}) \\ &\leq \dim \mathcal{L}_\lambda(\mathcal{A}) + \sum_{\mu\in\sigma(\mathcal{A})\setminus\{\lambda\}} m_1(\mu) = \dim \mathcal{L}_\lambda(\mathcal{A}) + M(\mathcal{A}) - m_1(\lambda). \end{aligned}$$

Hence (4.8) and (4.9) are proved. We continue with the proof of (4.10). Relation (4.11) implies

$$\sum_{\lambda\in\sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) \geq \sum_{\lambda\in\sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda) = n - \sum_{\lambda\in\sigma(\mathcal{A})} m_1(\lambda) = n - M(\mathcal{A})$$

and this yields

$$\sum_{\lambda\in\sigma(\mathcal{A}+\mathcal{P})\setminus\sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) = n - \sum_{\lambda\in\sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) \leq M(\mathcal{A}). \qquad \square$$

From the inequality (4.10) we see that the number of changeable eigenvalues under a rank one perturbation is bounded by $M(\mathcal{A})$.

5. Eigenvalue placement with rank one perturbations. In this section we study which sets of eigenvalues can be obtained by rank one perturbations. The following theorem is the main result of this note. It states that for a given set of complex numbers there exists a rank one perturbation $\mathcal{P}$ such that the set is included in $\sigma(\mathcal{A}+\mathcal{P})$, provided the given set has not more than $M(\mathcal{A})$ elements.

THEOREM 5.1. *Let $\mathcal{A}$ satisfy Assumption 2.2 and choose pairwise distinct numbers $\mu_1,\dots,\mu_l \in \overline{\mathbb{C}}$ with $l \le M(\mathcal{A})$. Choose multiplicities $m_1,\dots,m_l \in \mathbb{N}\setminus\{0\}$ with $\sum_{i=1}^{l} m_i = M(\mathcal{A})$. Then the following statements hold true.*

(a) There exists a rank one pencil $\mathcal{P}(s) = (\alpha s-\beta)uv^*$, $u, v \in \mathbb{C}^n$, such that $\mathcal{A}+\mathcal{P}$ is regular,

$$\sigma(\mathcal{A}+\mathcal{P}) = \{\mu_1,\dots,\mu_l\} \cup \{\lambda \in \sigma(\mathcal{A}) \mid \dim\ker \mathcal{A}(\lambda) \ge 2\} \tag{5.1}$$

and the multiplicities are given by

(5.2)

$$\dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) = \begin{cases} \dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda) + m_i, & \text{for } \lambda = \mu_i \in \sigma(\mathcal{A}),\\ \dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda), & \text{for } \lambda \in \sigma(\mathcal{A}) \setminus \{\mu_1,\dots,\mu_l\},\\ m_i, & \text{for } \lambda = \mu_i \notin \sigma(\mathcal{A}),\\ 0, & \text{for } \lambda \notin \sigma(\mathcal{A}) \cup \{\mu_1,\dots,\mu_l\}. \end{cases}$$

(b) Let in addition E, A be real matrices and $\{\mu_1,\dots,\mu_l\}$ symmetric with respect to the real line with $m_i = m_j$ if $\mu_j = \overline{\mu_i}$ and all $i, j = 1,\dots,l$. Then there exists $\alpha, \beta \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$ such that $\mathcal{P}(s) = (\alpha s - \beta)uv^T$ satisfies (5.1) and (5.2).

We formulate two special cases of Theorem 5.1.

COROLLARY 5.2. *In addition to the assumptions of Theorem 5.1 assume $k(\lambda) = 1$ for all $\lambda \in \sigma(\mathcal{A})$. Hence $m_1(\lambda) = \dim \mathcal{L}_\lambda(\mathcal{A})$ holds for all $\lambda \in \sigma(\mathcal{A})$ and $M(\mathcal{A}) = n$. Then there exists a rank one pencil $\mathcal{P}(s) = (\alpha s - \beta)uv^*$ such that the equations (5.1) and (5.2) take the following form*

$$\sigma(\mathcal{A}+\mathcal{P}) = \{\mu_1,\dots,\mu_l\} \quad \textit{and} \quad \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) = \begin{cases} m_i, & \text{for } \lambda = \mu_i,\\ 0, & \text{for } \lambda \notin \{\mu_1,\dots,\mu_l\}. \end{cases}$$

Therefore, for each $\mu_i \in \sigma(\mathcal{A}+\mathcal{P})$ there is only one Jordan chain of $\mathcal{A}+\mathcal{P}$ of length m_i.

If $E = I_n$ then we have $\sigma(\mathcal{A}) = \sigma(A)$ and especially $\infty \notin \sigma(\mathcal{A})$. Therefore the above Proposition 5.1 (a) leads for the choice $\alpha = 0$ and $\beta = 1$ to the following eigenvalue placement result for matrices.

COROLLARY 5.3. *For $A \in \mathbb{C}^{n\times n}$ with minimal polynomial m_A and values $\mu_1,\dots,\mu_l \in \mathbb{C}$ with multiplicities $m_i \in \mathbb{N}\setminus\{0\}$ satisfying $\sum_{i=1}^{l} m_i = \deg m_A$ there exists $u, v \in \mathbb{C}^n$ such that*

$$\sigma(A+uv^*) = \{\mu_1,\dots,\mu_l\} \cup \{\lambda \in \sigma(A) \mid \dim\ker(\lambda I_n - A) \ge 2\}$$

and (5.2) hold.

Before we prove Theorem 5.1, we show the following lemma.

LEMMA 5.4. *Let $\mathcal{A}$ satisfy Assumption 2.2. Then for the minimal polynomial m_J in (2.6) and for every polynomial p with complex coefficients and*

$$\deg p \le M(\mathcal{A}) - 1$$

there exist $u, v \in \mathbb{C}^n$ with

$$p(s) = v^* m_J(s)(sE - A)^{-1}u. \tag{5.3}$$

Given additionally that E and A are real valued and that p has real coefficients, then there exists $u, v \in \mathbb{R}^n$ satisfying (5.3).

Proof. We introduce

$$\Theta_{\mathcal{A}} : \mathbb{C}^n \times \mathbb{C}^n \to \big\{p \text{ polynomial} \mid \deg p \le M(\mathcal{A}) - 1\big\}, \quad (u, v) \mapsto v^* m_J(s)(sE - A)^{-1}u$$

and show the surjectivity of this map. Since the surjectivity of $\Theta_{\mathcal{A}}$ is invariant under basis transformations, we can assume that $\mathcal{A}$ is given in Weierstraß canonical form (2.3) with matrices J and N. If $\sigma(J) = \{\lambda_1, \ldots, \lambda_m\}$ for some complex numbers $\lambda_1, \ldots, \lambda_m$, then J and N are given by

$$J = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k(\lambda_i)} J_{m_j(\lambda_i)}(\lambda_i), \quad N = \bigoplus_{j=1}^{k(\infty)} J_{m_j(\infty)}(0) \tag{5.4}$$

with Jordan blocks $J_k(\lambda)$ of size k at $\lambda \in \mathbb{C}$. This allows us to simplify the resolvent representation with $u = (u_0^*, u_1^*)^*$, $v = (v_0^*, v_1^*)^*$, $u_0, v_0 \in \mathbb{C}^r$ and $u_1, v_1 \in \mathbb{C}^{n-r}$ to

$$v^* m_J(s)(sE - A)^{-1}u = v_0^* m_J(s)(sI_r - J)^{-1}u_0 + v_1^* m_J(s)(sN - I_{n-r})^{-1}u_1$$

$$= v_0^* m_J(s) \bigoplus_{\substack{i = 1, \ldots, m, \\ j = 1, \ldots, k(\lambda_i)}} \begin{pmatrix} (s-\lambda_i)^{-1} & \ldots & (-1)^{-m_j(\lambda_i)+1}(s-\lambda_i)^{-m_j(\lambda_i)} \\ & \ddots & \vdots \\ & & (s-\lambda_i)^{-1} \end{pmatrix} u_0 \tag{5.5}$$

$$+ v_1^* m_J(s) \bigoplus_{j=1,\ldots,k(\infty)} \begin{pmatrix} -1 & -s & \ldots & -s^{m_j(\infty)-1} \\ & -1 & \ldots & -s^{m_j(\infty)-2} \\ & & \ddots & \vdots \\ & & & -1 \end{pmatrix} u_1. \tag{5.6}$$

Observe $\mathcal{M}(\mathcal{A}) = \deg m_J + m_1(\infty)$. From (2.5) and (2.6) we see that $\Theta_{\mathcal{A}}$ maps into the set

$$\{p \text{ polynomial} \mid \deg p \le M(\mathcal{A}) - 1\}. \tag{5.7}$$

Obviously, for $j = 1$ and $i = 1, \ldots, m$ the entries of the first row of the blocks in (5.5) are linearly independent as they are functions with a pole in λ_i of order from one up to $m_1(\lambda_i)$. After multiplication with m_J this set of functions remains linearly independent. Therefore the set

$$P_1 := \{m_J(s)(s-\lambda_i)^{-r} \mid i = 1, \ldots, m, \ r = 1, \ldots, m_1(\lambda_i)\} \tag{5.8}$$

is linearly independent and contains $\sum_{i=1}^m m_1(\lambda_i) = \deg m_J$ elements, each of degree less or equal to $\deg m_J - 1$. Moreover, for $j = 1$, the entries of the first row of the blocks in (5.6) are linearly independent and form the linearly independent set of polynomials

$$P_2 := \{m_J(s)s^r \mid r = 0, \ldots, m_1(\infty) - 1\} \tag{5.9}$$

which contains $m_1(\infty)$ elements of degree between $\deg m_J$ and $\deg m_J + m_1(\infty) - 1 = \mathcal{M}(\mathcal{A}) - 1$. Hence $P_1 \cup P_2$ consists of

$$\deg m_J + m_1(\infty) = M(\mathcal{A}) \tag{5.10}$$

linearly independent elements. Furthermore by choosing one entry of u and v as one and all others as zero we see

$$P_1 \cup P_2 \subset \operatorname{ran} \Theta_{\mathcal{A}} \tag{5.11}$$

and with (5.10) Lemma 5.4 is proved for complex polynomials p and matrices E and A.
We consider the case where E, A are real valued. Here we use the Weierstraß canonical form over $\mathbb{R}$ (cf. [11]) with transformation matrices $S, T \in \mathbb{R}^{n\times n}$. The matrix N is the same as in (5.4) and J is in real Jordan canonical form (see, e.g., [15, Section 3.4.1]),

$$J = \bigoplus_{\substack{\lambda\,\in\,\sigma(J),\\ \operatorname{Im}\lambda\,>\,0}} \bigoplus_{j=1}^{k(\lambda)} J^{\mathbb{R}}_{m_j(\lambda)}(\lambda) \oplus \bigoplus_{\lambda\in\sigma(J)\cap\mathbb{R}} \bigoplus_{j=1}^{k(\lambda)} J_{m_j(\lambda)}(\lambda), \tag{5.12}$$

where $J_{m_j(\lambda)}(\lambda)$, $\lambda \in \sigma(\mathcal{A}) \cap \mathbb{R}$, are Jordan blocks of size $m_j(\lambda)$ and $J^{\mathbb{R}}_l(\lambda) \in \mathbb{R}^{2l\times 2l}$ is a real Jordan block at $\lambda = a + ib$ with $a \in \mathbb{R}$, $b > 0$, given by

$$J^{\mathbb{R}}_l(\lambda) := \begin{pmatrix} C(a,b) & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & C(a,b) \end{pmatrix} \in \mathbb{R}^{2l\times 2l}, \quad C(a,b) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Therefore the resolvent of $J^{\mathbb{R}}_l(a,b)$ is given by

$$(sI_{2l} - J^{\mathbb{R}}_l(a,b))^{-1} = \begin{pmatrix} (s - C(a,b))^{-1} & \dots & (-1)^{l-1}(s - C(a,b))^{-l} \\ & \ddots & \vdots \\ & & (s - C(a,b))^{-1} \end{pmatrix} \tag{5.13}$$

where the entries are given by

$$(s-C(a,b))^{-k} = ((s-C(a,b))^{-1})^k = \left(\frac{1}{(s-a)^2+b^2}\begin{pmatrix} s-a & b \\ -b & s-a \end{pmatrix}\right)^k, \quad k \in \mathbb{N}\backslash\{0\}.$$

Using the expression (5.13) instead of the blocks in (5.5) for the non-real eigenvalues, one can define again a linearly independent set of polynomials P_1 by picking all first row entries. This set consists again of polynomials all of distinct degree, because the factor $((s-a)^2 + b^2)^{m_1(\lambda)}$ occurs in the minimal polynomial m_J. The set P_2 remains the same as in the complex valued case. Therefore the same arguments imply the surjectivity of $\Theta_{\mathcal{A}}$ in this case. □

Proof of Theorem 5.1. Choose $\alpha, \beta \in \mathbb{C}$ such that

$$\begin{aligned} \frac{\beta}{\alpha} &\notin \{\mu_1, \dots, \mu_l\} \cup \sigma(\mathcal{A}), \quad \text{if } \alpha \neq 0 \text{ and} \\ \infty &\notin \{\mu_1, \dots, \mu_l\} \cup \sigma(\mathcal{A}), \quad \text{if } \alpha = 0 \end{aligned} \tag{5.14}$$

holds. In particular one can choose $\alpha \neq 0$ and we define

$$\gamma := \begin{cases} m_J(\beta/\alpha) \cdot \prod_{i=1}^{l} (\beta/\alpha - \mu_i)^{-m_i}, & \text{if } \infty \notin \{\mu_1, \dots, \mu_l\}, \\ m_J(\beta/\alpha) \cdot \prod_{i=1, i\neq j}^{l} (\beta/\alpha - \mu_i)^{-m_i}, & \text{if } \infty = \mu_j. \end{cases}$$

The condition $\beta/\alpha \notin \{\mu_1, \dots, \mu_l\} \cup \sigma(\mathcal{A})$ implies $m_J(\beta/\alpha) \neq 0$, hence $\gamma \neq 0$. We consider

$$q_\gamma(s) := \begin{cases} \gamma \prod_{i=1}^{l} (s - \mu_i)^{m_i}, & \text{if } \infty \notin \{\mu_1, \dots, \mu_l\}, \\ \gamma \prod_{i=1, i\neq j}^{l} (s - \mu_i)^{m_i}, & \text{if } \mu_j = \infty. \end{cases}$$

As $\sum_{i=1}^{l} m_i = M(\mathcal{A})$, the polynomial q_γ satisfies $\deg q_\gamma \leq M(\mathcal{A})$. The degree of m_J is $M(\mathcal{A}) - m_1(\infty)$ which is smaller or equal to $M(\mathcal{A})$. From the choice of γ we see for $\alpha \neq 0$ that $(q_\gamma - m_J)(\beta/\alpha) = 0$ holds and therefore

$$\frac{q_\gamma(s) - m_J(s)}{\alpha s - \beta} \tag{5.15}$$

is a polynomial of degree less or equal to $M(\mathcal{A}) - 1$. By Lemma 5.4 there exist $u, v \in \mathbb{C}^n$ with

$$\frac{q_\gamma(s) - m_J(s)}{\alpha s - \beta} = v^* m_J(s)(sE - A)^{-1} u. \tag{5.16}$$

This equation combined with Sylvester's determinant identity leads to

$$\frac{q_\gamma(s)}{m_J(s)} = 1 + (\alpha s - \beta) v^* (sE - A)^{-1} u = \det(I_n + (sE - A)^{-1} (\alpha s - \beta) u v^*), \tag{5.17}$$

for $s \notin \sigma(\mathcal{A})$. Now, set $\mathcal{P}(s) = (\alpha s - \beta) u v^*$, then from (5.17) we obtain for the pencil $\mathcal{A} + \mathcal{P}$ and $s \notin \sigma(\mathcal{A})$

$$\begin{aligned} \det(\mathcal{A}(s) + \mathcal{P}(s)) &= \det(sE - A + (\alpha s - \beta) u v^*) \\ &= \det(sE - A) \det(I_n + (sE - A)^{-1} (\alpha s - \beta) u v^*) \\ &= \det(sE - A) \frac{q_\gamma(s)}{m_J(s)}. \end{aligned} \tag{5.18}$$

Since $\det(sE - A)$ is by Proposition 2.1 (c) divisible by m_J this implies (5.1). Moreover, the characteristic polynomial of $\mathcal{A} + \mathcal{P}$ is unequal to zero, hence the pencil $\mathcal{A} + \mathcal{P}$ is regular. The equation (5.2) follows from Proposition 2.1 (c) and Theorem 5.1 (a) is proved. The statements in (b) follow by the same construction as above and by Lemma 5.4 for real valued matrices. □

REMARK 5.5. *The construction in the proof of Theorem 5.1 above showed that in particular for any non-zero pair $(\alpha, \beta) \in \mathbb{C}^2$ satisfying* (5.14) *one can construct $u, v \in \mathbb{C}^n$ such that $\mathcal{P}(s) = (\alpha s - \beta) u v^*$ satisfies* (5.1) *and* (5.2).

6. Eigenvalue placement under parameter restrictions. In this section we study the eigenvalue placement under perturbations of the form

$$\mathcal{P}(s) = (su + v) w^*, \quad w \in \mathbb{C}^n \tag{6.1}$$

where now u and v are fixed elements of $\mathbb{C}^n$. A special case of this placement is the feedback stabilization problem (cf. [6]). Obviously, for the perturbations (6.1) the bounds on the multiplicities from Theorem 4.4 still hold. But since u and v are now fixed, we obtain tighter bounds which are given below.

PROPOSITION 6.1. *Let $\mathcal{A}(s) = sE - A$ be regular and $\mathcal{P}$ be given by* (6.1) *with $u, v \in \mathbb{C}^n$ fixed. The function $s \mapsto (sE - A)^{-1}(su + v)$ has a pole at $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$. Denote the order of this pole by $m_{uv}(\lambda)$ and denote by $m_{uv}(\infty)$ the pole order of $s \mapsto (-sA + E)^{-1}(sv + u)$ at zero. Then the following inequalities hold.*
For $\lambda \in \sigma(\mathcal{A})$ we have

$$\dim \mathcal{L}_\lambda(\mathcal{A}) - \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq m_{uv}(\lambda),$$

and with $\mathcal{M}(\mathcal{A}, u, v) := \sum_{\lambda \in \sigma(\mathcal{A})} m_{uv}(\lambda)$ and $\lambda \in \sigma(\mathcal{A})$

$$\dim \mathcal{L}_\lambda(\mathcal{A}) - m_{uv}(\lambda) \leq \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq \dim \mathcal{L}_\lambda(\mathcal{A}) + \mathcal{M}(\mathcal{A}, u, v) - m_{uv}(\lambda)$$

holds. For $\lambda \in \overline{\mathbb{C}} \setminus \sigma(\mathcal{A})$ we have

$$0 \leq \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq \mathcal{M}(\mathcal{A}, u, v).$$

From this we obtain

$$\sum_{\lambda \in \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \geq n - \mathcal{M}(\mathcal{A}, u, v),$$
$$\sum_{\lambda \in \sigma(\mathcal{A}+\mathcal{P}) \setminus \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq \mathcal{M}(\mathcal{A}, u, v).$$

Proof. We show for all $\lambda \in \sigma(\mathcal{A})$

$$\dim \mathcal{L}_\lambda(\mathcal{A}) - m_{uv}(\lambda) \leq \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}).$$

From Sylvester's formula we conclude

$$\det(\mathcal{A}(s) + \mathcal{P}(s)) = \det \mathcal{A}(s)(1 + w(sE - A)^{-1}(su + v)).$$

The characteristic polynomial of $\mathcal{A}$ has at $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$ a zero of multiplicity $\dim \mathcal{L}_\lambda(\mathcal{A})$. Since the pole order of $1 + w(sE - A)^{-1}(su + v)$ is by definition $m_{uv}(\lambda)$, the zero multiplicity of $\det(\mathcal{A} + \mathcal{P})(s)$ at λ, hence the dimension of $\mathcal{L}_\lambda(\mathcal{A} + \mathcal{P})$, can be bounded by

$$\dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \geq \dim \mathcal{L}_\lambda(\mathcal{A}) - m_{uv}(\lambda).$$

For $\lambda = \infty$ this inequality can be derived from the fact that the root subspace of $sE - A$ at ∞ can be identified with the root subspace dual pencil $-sA + E$ at 0 (cf. [8, 9]). The remaining inequalities can be obtained in the same way is in the proof of Theorem 4.4. □

Within these bounds we investigate the placeability of the eigenvalues.

THEOREM 6.2. *Let $\mathcal{A}(s) = sE - A$ satisfy Assumption 2.2 and let $\mathcal{P}$ be given by* (6.1) *with $u, v \in \mathbb{C}^n$ fixed. Given $\mu_1, \ldots, \mu_l$ with $l \leq \mathcal{M}(\mathcal{A}, u, v)$ and multiplicities $\sum_{i=1}^{l} m_i = \mathcal{M}(\mathcal{A}, u, v)$ then we obtain the following.*

(a) *For every $\mathcal{P}(s) = (su + v)w^*$ with $w \in \mathbb{C}^n$ we have*

$$\{\lambda \in \sigma(\mathcal{A}) \mid k(\lambda) \geq 2 \text{ or } m_{uv}(\lambda) < m_1(\lambda)\} \subseteq \sigma(\mathcal{A} + \mathcal{P}).$$

(b) *Assume that u and v are linearly dependent, i.e. $\mathcal{P}(s) = (\alpha s - \beta)vw^*$ or $\mathcal{P}(s) = (\alpha s - \beta)uw^*$ for $\alpha, \beta \in \mathbb{C}$. Then we have the following cases.*

(b.1) *For $\beta/\alpha \in \sigma(\mathcal{A})$ there exists $w \in \mathbb{C}^n$ such that*

$$\sigma(\mathcal{A} + \mathcal{P}) = \{\mu_1, \ldots, \mu_l\} \cup \{\lambda \in \sigma(\mathcal{A}) \mid k(\lambda) \geq 2 \text{ or } m_{uv}(\lambda) < m_1(\lambda)\}$$

holds.

(b.2) *For $\beta/\alpha \notin \sigma(\mathcal{A})$ there exists $w \in \mathbb{C}^n$ such that the equation in (b.1) holds, if and only if $\beta/\alpha \notin \{\mu_1, \ldots, \mu_l\}$.*

(c) *Assume that u and v are linearly independent, then there exists $w \in \mathbb{C}^n$ such that $\mathcal{P}(s) = (su + v)w^*$ satisfies*

$$\sigma(\mathcal{A} + \mathcal{P}) = \{\mu_1, \ldots, \mu_l\} \cup \{\lambda \in \sigma(\mathcal{A}) \mid k(\lambda) \geq 2 \text{ or } m_{uv}(\lambda) < m_1(\lambda)\}.$$

The multiplicities under these perturbations are given by

$$\dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) = \begin{cases} \dim \mathcal{L}_\lambda(\mathcal{A}) - m_{uv}(\lambda) + m_i, & \text{for } \lambda = \mu_i \in \sigma(\mathcal{A}), \\ \dim \mathcal{L}_\lambda(\mathcal{A}) - m_{uv}(\lambda), & \text{for } \lambda \in \sigma(\mathcal{A}) \setminus \{\mu_1, \ldots, \mu_l\}, \\ m_i, & \text{for } \lambda = \mu_i \notin \sigma(\mathcal{A}), \\ 0, & \text{for } \lambda \notin \sigma(\mathcal{A}) \cup \{\mu_1, \ldots, \mu_l\}. \end{cases} \tag{6.2}$$

Proof. From Corollary 4.3 we already know that the $\lambda \in \sigma(\mathcal{A})$ with $k(\lambda) \geq 2$ are contained in the spectrum of $\sigma(\mathcal{A} + \mathcal{P})$. We consider only the case $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$. The resolvent representation (5.5) implies that the pole order of $s \mapsto (sE - A)^{-1}$ at $\lambda \in \sigma(\mathcal{A})$ is $m_1(\lambda)$. Hence $m_{uv}(\lambda) \leq m_1(\lambda)$ and $m_{uv}(\lambda) < m_1(\lambda)$ implies that all the polynomials in (5.8) have the common factor $(s - \lambda)$. Hence $\lambda \in \sigma(\mathcal{A} + \mathcal{P})$.
Now for the placement results one can just use the arguments from above with $\mathcal{M}(\mathcal{A})$ replaced by $\mathcal{M}(\mathcal{A}, u, v)$. In particular, reconsidering the essential Lemma 5.4, then instead of the linear independent set of polynomials (5.8) and (5.9) one uses

$$P_1 := \{\widetilde{m}_J(s)(s - \lambda)^{-j} \mid 1 \leq j \leq m_{uv}(\lambda)\}, \quad \widetilde{m}_J(s) := \prod_{\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}} (s - \lambda)^{m_{uv}(\lambda)},$$
$$P_2 := \{\widetilde{m}_J(s)s^j \mid 0 \leq j \leq m_{uv}(\infty) - 1\}.$$

The set $P_1 \cup P_2$ is a basis in the space of polynomials of degree less or equal to $\mathcal{M}(\mathcal{A}, u, v)$. For the proof of (b) we use the equation (5.16). Here we see that $\beta/\alpha \notin \sigma(\mathcal{A})$ implies $m_J(\beta/\alpha) \neq 0$. Hence the characteristic polynomial of $\mathcal{A}+\mathcal{P}$ must satisfy $\det(\mathcal{A} + \mathcal{P})(\beta/\alpha) \neq 0$. Therefore this eigenvalue can not be obtained under a rank one perturbation. For (c) the construction of the sets P_1 and P_2 can be carried out as above. ◻

Parameter restricted perturbations of the form (6.1) occur naturally in the study of differential algebraic equations with a single input given by $E, A \in \mathbb{C}^{n \times n}$, $b \in \mathbb{C}^n$ and the equation

$$\frac{d}{dt}Ex(t) = Ax(t) + bu(t), \quad t \in \mathbb{R}. \tag{6.3}$$

For this equation we consider a state feedback of the form $u(t) = f^*x(t)$ with $f \in \mathbb{C}^n$. It is well known that the solution of the closed loop-system

$$\frac{d}{dt}Ex(t) = (A + bf^*)x(t), \quad t \in \mathbb{R}. \tag{6.4}$$

can be expressed with the eigenvalues and Jordan chains of the matrix pencil $sE - (A + bf^*)$ (cf. [5]). This pencil can be written as a perturbation of $sE - A$ with the rank one pencil $\mathcal{P}(s) = -bf^*$. By fixing $u = 0$ and $v = -b$, we are in setting of Theorem 6.2 (b). Note that these feedback placement problems were studied in a more general form for singular matrix pencils in [6]. But for single input systems their conditions can be simplified.

THEOREM 6.3. *Let (E, A, b) be given by (6.3) such that $\mathcal{A}(s) = sE - A$ is regular and E is singular. Choose pairwise distinct numbers $\mu_1, \dots, \mu_l \in \mathbb{C}$ with $l \leq \mathcal{M}(\mathcal{A}, 0, -b)$. Choose multiplicities $m_1, \dots, m_l \in \mathbb{N}\backslash\{0\}$ with $\sum_{i=1}^{l} m_i = \mathcal{M}(\mathcal{A}, 0, -b)$ then there exists a feedback $f \in \mathbb{C}^n$, such that $sE - (A + bf^*)$ is regular with*

$$\sigma(sE - (A + bf^*)) = \{\mu_1, \dots, \mu_l\} \cup \{\lambda \in \sigma(\mathcal{A}) \mid \dim \ker \mathcal{A}(\lambda) \geq 2 \text{ or } m_{0-b}(\lambda) < m_1(\lambda)\}.$$

The multiplicities under this perturbation are given by (6.2).

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